

Representations in problem solving: a case study with optimization problems¹

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Abstract

Introduction. Representations play an essential role in mathematical thinking. They favor the understanding of mathematical concepts and stimulate the development of flexible and versatile thinking in problem solving. Here our focus is on their use in optimization problems, a type of problem considered important in mathematics teaching and learning in higher education.

Method. By using an observational methodology, we present the representation patterns used by three students in their fifth year of an undergraduate mathematics degree at the University of Granada (Spain). The patterns were obtained through protocol analysis in which students' written production was complemented with their spoken thinking protocols. The instrument used was prepared intentionally for this study and was composed of three optimization problems. Students solved the instrument individually in an isolated environment while being video-recorded.

Results. We designed a framework for protocol analysis and used it to analyze the transcriptions of students' productions. Results are presented in the form of a microscopic analysis of the particular cases, with detailed records of the representations used by the students, translation between records and the time used in these actions. These elements are then used to define a profile of each Problem Solver, which was found to be different for the three cases.

Discussion and conclusions. The framework designed for the microscopic analysis of problem solving protocols was useful for describing the representations and the translations between them during the problem-solving process. The segmentation and coding process made evident the need to include episodes which initially were not identified as representations. A characterization of the Problem Solvers shows a strong connection between students' success in solving optimization problems and their skill in using representations.

Keywords: problem solving, representations, optimization problems, case studies.

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Resumen

Introducción. Las representaciones juegan un papel fundamental en el pensamiento matemático, favorecen la comprensión de los conceptos matemáticos y estimulan el desarrollo de un pensamiento flexible y versátil en la resolución de problemas. En este trabajo nos ceñimos a problemas de optimización, de gran importancia y predicamento en la enseñanza y aprendizaje de la matemática a nivel superior

Método. Mediante metodología observacional presentamos los patrones de representación empleados por tres estudiantes de quinto curso de la licenciatura de matemáticas de la Universidad de Granada (España), obtenidos mediante el análisis de protocolos, en el que los registros escritos van acompañados de protocolos de pensar en voz alta. El instrumento empleado fue construido *ad hoc* y consta de tres problemas de optimización. Las sesiones de resolución de problemas fueron individuales, en un ambiente aislado y fueron grabadas en vídeo.

Resultados. Para el estudio de los datos se diseñó un marco para el análisis de protocolos con el que se investigan las transcripciones de las producciones de los sujetos. Los resultados se exponen en forma de análisis microscópico de caso en los que se pormenorizan los registros de representación empleados, las traducciones entre registros y el tiempo empleado en estas acciones, a partir de los cuales se realiza un perfil diferenciador de los resolutores. Según este perfil los tres participantes tienen una tipología distinta.

Discusión y Conclusiones. El marco para el análisis microscópico de los protocolos de resolución de problemas se ha mostrado adecuado para describir las representaciones y traducción entre representaciones en resolución de problemas. El proceso de segmentación y codificación nos ha llevado a considerar necesario incluir episodios calificados en principio como eventos no catalogables como representaciones. La caracterización de los resolutores muestra la fuerte relación entre el éxito en la resolución de problemas de optimización y la habilidad en el manejo de las representaciones.

Palabras Clave: Resolución de problemas, representaciones, problemas de optimización, estudio de casos

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Introduction

One characteristic of human intelligence is the use of different types of representations, whether for recreational, normative, communicative, symbolic, artistic, literary or musical purposes. This characteristic differentiates us from animals and from artificial intelligence, and is perhaps one of the reasons that accounts for the recent proliferation of research on the place of representations in learning mathematics and in problem solving. As a result of this research, the importance of multiple representations in the development of mathematical thinking has become unquestionable (Brenner *et al.* 1997; Cuoco & Curzio, 2001). This can be amply observed in the priorities established by committees and at scientific meetings with an international scope (Goldin, 1998b; Hitt, 2002).

The term representation is complex, and is open to many interpretations (Rico, 2000). In this study, the term representation refers to the forms with which we present to ourselves mathematical objects and processes, and which we find essential for defining, explaining, visualizing, recording and communicating mathematical knowledge.

Representation systems fulfill certain requirements for complexity, interrelationship and power of symbolization and abstraction; mastering them broadens and enriches human intelligence, in that they are useful instruments for modeling reality and practical tools for solving different problems in real life. Through diverse means of expression, we as human beings learn and become familiar with an endless number of codes, symbols, signs, icons and languages, of many different types. The heuristic and communicative power that these representative elements make available in human activity is increased to the extent that these means of expression are formally integrated in complex symbolization systems which are subject to syntactic and grammatical rules.

The importance of representations is made evident in work by Duval (1998), for whom it is impossible to study knowledge-related phenomena without recourse to the notion of representation: no knowledge can be mobilized by a subject without activating representation. Thus, representations in mathematics and in mathematics education are fundamental; the object of study are mental constructions, and we require representations in order to interact with them.

In this paper we describe representations used by a group of university students when they solve a specific type of optimization problem where there are multiple possibilities for representing and modeling situations. Think-aloud protocols were used, and an empirical framework inspired by Schoenfeld's work (1985) was constructed in order to analyze them. This framework was applied in this study to analyze representation strategies, the time invested in each representation, and mechanisms for translating from one system to another.

The importance of representations in solving problems

Over the last two decades, a high level of consensus has been reached in the research community about students' use of representations, and their usefulness as an instrument at the service of problem solving (Castro, 2008; DeBellis & Goldin, 2006). Moreover, multiple representations can be used in order to develop the comprehension of concepts and processes in more depth and more flexibly (Cuoco & Curcio, 2001; Hiebert & Carpenter, 1992; Kaput, 1987; Koedinger & Nathan, 2004). While it is important to possess various representations of a concept, the mere existence of these is not enough to enable flexible use of the concept in solving problems. In order to successfully manage information used in problem solving, the representations must be correct and be strongly linked to each other: "One needs the possibility to switch from one representation to another one, whenever the other one is more efficient for the next step one wants to take" (Dreyfus, 1991).

Given the importance of problem solving in education (NCTM, 1989, 2000, 2006), the study of representations which students form when they solve problems is a topic of interest from both the educational and research perspectives. The National Council of Teachers of Mathematics (NCTM) recognizes that proper use of multiple representations contributes a flexible set of tools for problem solving and for appreciating the consistency and the beauty of mathematics. One objective of teachers in middle- and higher-level education is that students reach a good understanding of the different ways of representing, that they are able to articulate these representations without contradictions and that they turn to them spontaneously during problem solving, since this is key and essential to their success (Hitt, 1996).

Along these lines, Cifarelli (1998) states that the success of competent problem solvers may be due largely to their ability to construct appropriate representations for problem-solving situations. The very process of choosing the appropriate representations gives students

the chance to practice weighing the advantages and disadvantages of the different forms of representation (Schultz & Waters, 2000), and to use them as tools in problem solving. For their part, Lesh, Post and Behr (1987) describe the role played by representations, and translating between representations, in the learning of mathematics and in problem solving. They use the term representation in a restricted sense, as external expressions of the students' internal conceptualizations. For these researchers, the different representation systems are not only important "on their own merit", but also the *translations* between them and *transformations* within them are also important; thus, skill in translation (or lack of it) is a significant factor that affects both learning and performance in problem solving.

Duval (1993, 1998, 2006) underscores the importance of the activity of translating from one representation system to another, starting with word problems in a narrative format; he stresses the need for more research in this area to be carried out. All the tasks selected for the study presented here meet the above recommendations, namely: they are word problems whose solution falls within the realm of mathematics, they allow for the use of more than one representation system as a feasible channel for structuring, visualizing and searching for the solution. The tasks also belong to a specific typology of mathematics word problems, referred to as optimization.

Optimization problems. Reasons for their use in this study.

In previous studies on representations and problem solving, a strong association is found between the types of representations which problem-solvers use and the type of task that was proposed (Castro, *et al.*, 1999). It is appropriate for this reason to carry out studies with groups of problems that belong to a single conceptual unit of mathematic content. In this study we focus on optimization problems, which make up an important segment in mathematics education.

The desire for optimization is inherent to humans. The search for extremes inspires mountain-climbers, scientists, mathematicians and the human being in general. Methods of optimization explore suppositions about the nature of responses to the target function, by varying parameters and suggesting the best way to change them. The variety of *a priori* suppositions corresponds to the variety of optimization methods.

In order to explain how difficult optimization problems are, we can observe their classification in Figure 1. Many optimization methods are designed for a diverse number of independent controls (dimensionality). These ranks correspond to one or more variables.

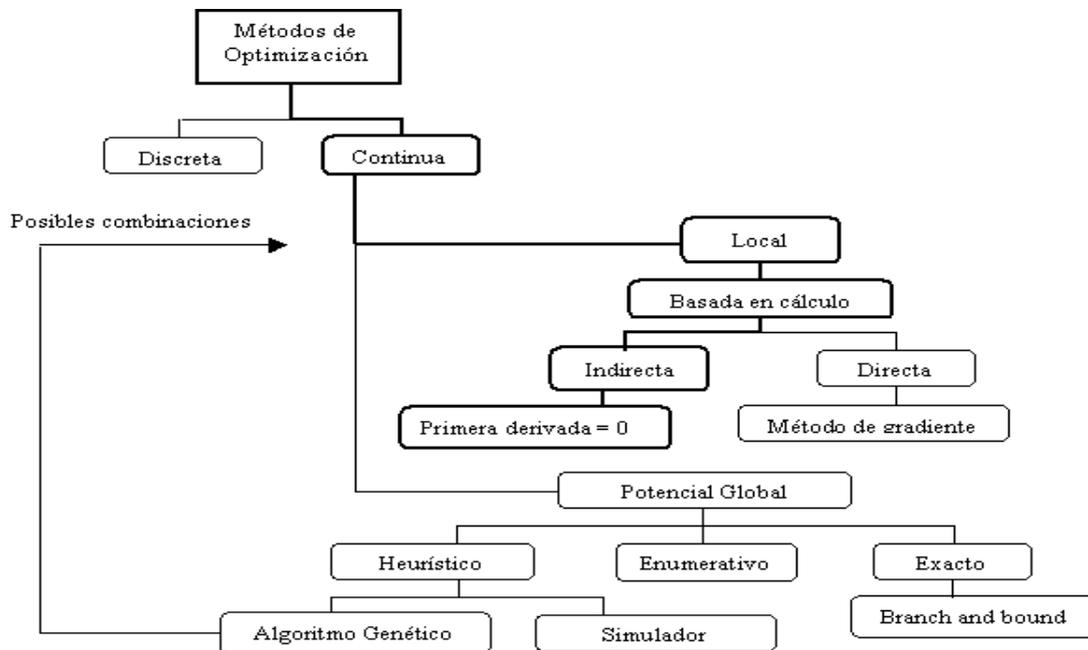


Figure 1. Optimization tree

The optimization problems we will use in this study follow the path marked in bold in Figure 1, where a single independent variable intervenes. This type of problem is among the most frequent and noteworthy applications of calculus; furthermore, daily life is full of practical applications where these types of problems are involved. For these reasons, we have selected this type of problem for our study: because it constitutes a well-delimited field of mathematical problems from the conceptual point of view, because it is highly applicable and because of its intrinsic potential for using several forms of representation in problem solving.

The specific optimization problems selected are word problems related to the real world. Thus, the initial representation is a word problem or text which must be translated in order to construct a mathematical structure; this representation process has particular importance in applied mathematics, being emphasized at university and more recently in the school curriculum, under the name of modeling. The term modeling refers to the search for a mathematical representation for a non-mathematical object or process; in this case, it means constructing a mathematical structure or theory which incorporates essential characteristics of

the object, system or process to be described. This model-structure may in turn be used to study the behavior of the object or process that is being modeled (Dreyfus, 1991).

The process of representing, to a certain degree, is analogous to the process of modeling, but it takes place at a different level: in modeling the situation or system being worked with is physical, and the model is mathematical; in representation, the object to be represented is the mathematical structure, and the model is a mental structure. Consequently, the mental representation is related to the mathematical model as the mathematical model is related to the physical system (Dreyfus, 1991). Each one is a partial translation of the other, each one reflects certain properties, but not all, of the other's; and each one increases the capacity to mentally manipulate the system under consideration.

Optimization problems have been used in many studies related to representations and problem solving (Camacho & González, 1998; Campos & Estrada, 1999; Porzio, 1999; Shoenfeld, 1985). Likewise, this type of problem is quite important in the mathematics curriculum of the later years of secondary education and the early years of university degrees in science, engineering, economics and finance.

The value and educational interest of optimization problems has been widely recognized in the region of Andalusia (Junta de Andalucía, 1989) in Mathematics Curriculum Plans for students between 16 and 18 years of age, where the great value of optimization problems in the teaching and learning of mathematics is noted for the following reasons:

- They are applicable and useful for analyzing and finding solutions for very diverse practical matters.
- They make evident the power of differential calculus, providing a general method for addressing extreme problems.
- They elicit very suitable situations, from the educational point of view, for practicing heuristics as a valuable procedure in solving problems.
- They produce added motivation, due to their potential for application, and the creativity required in using multiple representations.

This type of problem has been widely used in situations that apply the concepts of calculus, and directed toward advanced levels of conceptual mastery of mathematical aspects.

Using representation systems in these spheres of mathematics education broadens the possibilities of working at lower levels that do not require expert mastery of the content.

Objectives of the study

The general objective of this study is to describe the representations, and processes of translating between representations, used by a group of university students in solving optimization problems.

Among the specific objectives which make up the overall study, we highlight the following:

1. To determine what type of representations are used by expert problem solvers in solving optimization problems.
2. To evaluate how the time spent on each representation and on translation between representations affects the success of problem solvers in solving optimization problems.
3. To look for any regularities which may exist in the use of representations or in translating between representations when solving optimization problems.
4. To characterize problem solvers according to the representations and translations between representations that they use in different problem situations.

For this purpose, we took into account the following types of external representations:

1. Verbal representation of the word problem: consisting fundamentally of the word problem as stated, whether in writing or spoken;
2. Pictorial representation: consisting of drawings, diagrams or graphs, as well as any kind of related action;
3. Symbolic representation: being made up of numbers, operation and relation signs; algebraic symbols, and any kind of action referring to these;

in addition to translations between all the above (Fig. 2).

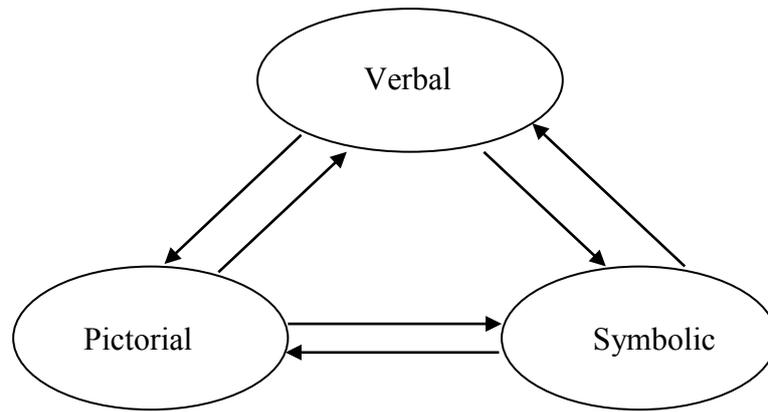


Figure 2. Types of representation systems

An initial difficulty of the study, in order to meet the overall objective, was the lack of a framework for analyzing protocols in solving optimization problems, one which would take into account the representations and translations between representations. For this reason it was necessary to develop such a framework, which we did by adapting the protocol analysis prepared by Schoenfeld (1985).

In particular, our interest focused on describing the differences and similarities in the variable of time spent on each representation and in translation between representations, as a function of the type and profile of the problem solver, as well as to assess the representational potential of the different types of optimization problems selected, and the versatility of different expert students in solving them.

This study belongs to the tradition that considers certain dimensions of representations to be determining factors in problem solving (Ballard, 2000; Cifarelli, 1998; Janvier, 1987; Lesh, Post & Behr, 1987; Porzio, 1999).

Method

Participants

This study can be considered an intrinsic case study in the terms defined by Stake (1999) and Gutiérrez *et al.* (2002): a study of the particularity and the complexity of one or several unique cases, in order to understand their activity under determined circumstances;

and with the meaning given by Marcelo (1991), as “an intensive or comprehensive examination of one facet or dimension over time”. The subjects being observed in this study were three students in the fifth year of a Mathematics degree with a specialty in Methodology, at the Faculty of Sciences, University of Granada, academic year 2001-2002.

Instruments

The study follows the guidelines of observational methodology (Anguera *et al*, 1995), in line with recommendations given by Dane (1990) regarding the observational method as a systematic research procedure whereby events are selected, recorded and coded in significant analysis units. Specifically, the technique used for data collection was protocol analysis, taking protocol to mean the chronologically ordered description of a subject’s behaviors while he or she executes a problem-solving task, in which the data to be analyzed are the verbalizations made during problem solving, as well as the subject’s notes, representations, sequencies and strategies deployed in the solution process, accompanied by a few comments and observations from the researcher (Schoenfeld, 1985). Thus, we are interested in analyzing not only the semiotic representations which the student expresses explicitly, but also the mental representations inferred in speech or through observable behavior. The subject is asked to use the thinking aloud technique, which provides a great increase in the amount of behavior which can be observed, as compared to the same subject working in silence. Moreover, it is not necessary for the subject to have previous training in order to be able to think aloud on this type of problem-solving task, and it can be inferred that this verbal report is consistent with the structure of his or her normal cognitive processes (Ericsson & Simon, 1993).

Construction and organization of the problem solving session

Our first step, faced with the need for a confirmed instrument for data collection, was to carry out a pilot study that made it possible to prepare our final questionnaire. At the same time we wanted to examine other important parameters, such as the time needed for solving the problems, the order and manner of presenting the problems; the technology that would best suit our purposes for data collection during the problem-solving session, and any other aspects that might be useful to consider in the final study.

First, a battery of optimization problems was built, using the following criteria: the problems were found in university text books, they were suitable for students in the first year of a university degree in science or engineering, problems were strictly “narrative” in format, they were mathematics problems from the real world, use of calculators was not necessary, and use of several representations was needed in order to reach a solution.

A set of 47 problems that met the above requirements was collected; these were submitted for consideration to several researchers and mathematics graduates, asking their opinion about the pertinence of these problems to our proposed study, and if they were adequate for the subjects who were to participate. Based on these opinions, four problems were selected for use in a pilot study, and applied to three volunteers who had graduated in mathematics between 2 and 6 years before. They solved the problems while thinking aloud, and the problem-solving sessions were recorded on audio.

From the results of the pilot test application, the number of problems was reduced to three, since one of the problems was considered “complicated” and incongruent with the others. It was not necessary to substitute this problem, since four problems were found to be excessive in order to be solved in a single session. The time required for each problem was established at 20 minutes, bringing the total time to approximately 90 minutes for each participant: 60 minutes for solving the problems and 30 minutes for the solver to make final comments about each problem and about the process in general. We also found it to be absolutely necessary for the sessions to be recorded on video, with additional support from a tape recording, which would reinforce the sound from the video recorder.

Taking into consideration the information gained through the pilot test, the problems chosen for the final study were the following:

Table 1. Problems Posed

PROBLEM 1: *An electrical station is located on one side of a straight river, one kilometer wide. Five kilometers upstream, on the other side of the river, there is a factory. If the owner wishes to lay a cable from the electrical station to the factory, he knows that laying cable underground costs 3 euros per meter, and laying it underwater costs 5 euros per meter. What would be the most economical route for the cable to be laid? And if underground cable cost the same as underwater cable, what would the route be?*

PROBLEM 2: *We want to build a window in the shape of a rectangle topped by a semi-circle (the width of the rectangle must be equal to the width of the semi-circle). What type of window would let in the most possible light, if the perimeter must be fixed?*

PROBLEM 3: *A rectangular mirror measures 80 by 90 centimeters. A corner of the mirror breaks off on a straight line. Of the two pieces that remain, the small one is a right triangle with 10- and 12- cm. cathetus, corresponding to the short and long sides of the mirror, respectively. What is the largest rectangular mirror that can be obtained from the larger piece?*

The following characteristics apply to the problems selected:

Table 2. Problem characteristics

	Type of problem	Type of optimization	Functions involved	Numbers involved
Problem 1	By two means	Minimal cost	Linear Radical	Large integers Fractions
Problem 2	Minimize construction	Minimum area	Linear Quadratic	Fractions Irrational
Problem 3	Inscribe/ circumscribe	Maximum area	Linear Quadratic	Large integers Fractions

Procedure

The students had to solve the three optimization problems presented above. The problems were distributed to the subjects one at a time, and they were not given the next problem until they had finished and turned in the previous one. Each problem was presented on a single sheet of paper, in large type, in order to enhance observation of the problems and data transcription.

The video-recording sessions began at 9:30 AM, with a minimal number of persons present in the room. In addition to the problem-solver, only the researcher and a research as-

sistant were present, the latter in charge of helping with the video and audio equipment. The purpose of limiting the number of persons present during the test was to minimize any influence on the problem solver's behavior. The researcher and the assistant acted as observers, taking notes on what they considered to be significant, and on aspects which they felt would not be reflected in the video recording.

The problems presented were solved while thinking aloud, with the least possible intervention from the researcher during the problem solving process. In order to obtain and describe activities that constitute problem solving, the data must be obtained from a situation where the subject is dealing with stimulating problems (in our case, optimization problems) and where there is minimal intervention from the researcher (Gindburg *et al.*, 1983). A few interventions did occur when the problem solver remained silent for a long time, or was "going round and round" the problem without making any progress, and the researcher considered that it was of more value to the study to "point the solver in the right direction", thus triggering the production of more information for the study.

Before initiating the problem-solving process, the researcher informed the subject about the following matters related to the investigation and to the problem-solving process: the study's future importance to the teaching and learning process; why he or she was invited to participate in the study; that they would be recorded on video while solving the problems; the approximate time involved in the tasks; the importance of taking the activity seriously; the need to think aloud from start to finish while working on the assigned problems; the need to verbalize all steps while solving the problem, without assuming that some thought or step was too obvious or trivial to verbalize; to not erase anything that had been written, but if necessary, to cross it out; that the results on these tasks would have no repercussions on their academic grades for any course they were taking at the university; and that the information they supplied would be strictly confidential.

By talking with the solvers about all of this, besides letting them know how to act while solving the problems, we also sought to establish rapport between solver and researcher, to set the solver at ease, so that his or her state of mind would affect the problem-solving process as little as possible.

Data Analysis

In order to produce the transcriptions, the researcher first watched the video carefully, without taking any written notes, in order to become familiarized with the solver's voice and gestures; afterward a first transcription was made, producing a printed text. A second researcher likewise made a transcription and a printed text. The two texts were compared and a joint version was produced, by watching the video again and correcting whatever was necessary in order to obtain the final transcription.

After this transcription, the text was segmented into different items. As a first criteria, each item corresponds to verbalizations that were produced without interruptions; a new item begins each time there was an interruption due to stalling, writing silently, asking questions of the researchers, or any other act that constituted an interruption in the course of problem solving. This initial division is already an analysis, since it is necessary to decide which interruptions will be taken into account in fragmenting the oral discourse, not to mention that merely placing punctuation marks in the transcription, so that it generally fits grammatical rules, is an analysis of the oral text produced by the subjects (Puig, 1996).

An adaptation of the protocol analysis designed by Schoenfeld (1985) was used as a basis for our study. Schoenfeld divides problem-solving protocols into macro-scopic pieces of constant behavior called episodes. An episode is a period of time during which a solver or group of solvers is occupied in a specific action. This definition of episode, in order to make sense, must be accompanied by a corresponding definition of what types of different behaviors will be used in order to classify and segment the protocol in pieces. In our analysis, instead of examining students' behavior in solving problems, as did Schoenfeld, we observed students' use of representations and of translation between representations when solving optimization problems. When we speak of using representations, we must be aware of what representations we are going to use, so, for our analysis we take three types of representations into account: the verbal representation of the word problem, pictorial representations and symbolic representations, as well as translations between them (see Fig. 2): from verbal to pictorial or viceversa, from pictorial to symbolic or viceversa, and from symbolic to verbal or viceversa.

Each one of these representations and the translations between representations constitutes an episode; in other words, there were six episodes. These episodes establish a second criterion for protocol analysis; to make this analysis, the prior transcriptions were reviewed, marking vertical lines in any paragraph that was considered to belong to a particular episode.

When making this classification, there were paragraphs that did not correspond to any of the six episodes above, so it was necessary to add a seventh episode, to include the events that were not catalogued as representations. It is important to note that the episodes are not linear in nature, that is, a problem solver uses one episode in particular at any moment and can be “skipping” between episodes while solving the problem.

In order to facilitate division of the written protocol, indicators were attached to the episodes described above, such that the episode categories are described theoretically and empirically (Artzt & Armour-Thomas, 1992). The framework for analysis of problem-solving protocols, which we built in order to analyze representations and translation between representations (Villegas & Castro, 2002), is made up of the episodes shown in Table 3.

Table 3. Framework for protocol analysis

<p>Episode 1: Internal representation of the word problem.</p> <p>Description: The student reads the problem.</p> <p>Indicators:</p> <ol style="list-style-type: none"> a. The student reads the problem aloud. b. The student reads the problem silently, or “mumbling”. c. The student verbalizes the problem, changing some words for others from his usual style of talk.
<p>Episode 2: Pictorial representation.</p> <p>Description: The student produces, operates with or modifies pictorial representations.</p> <p>Indicators:</p> <ol style="list-style-type: none"> a. The student draws a pictorial representation with pencil and paper, or modifies such representations made earlier. b. The student operates with pictorial representations. c. The student points to or observes a pictorial representation, or verbalizes terms associated with pictorial representations. d. The student uses body movements, whether with the hands or other parts of the body, to show pictorial representations.

Episode 3: Symbolic representations.

Description: The student produces, operates with or modifies symbolic representations.

Indicators:

- a. The student solves or tries to solve a symbolic expression with paper and pencil.
- b. The student verbalizes how he can solve an equation, or checks how it was solved.
- c. The student modifies, re-writes or eliminates a symbolic expression.
- d. The student observes or points to a symbolic expression.

Episode 4: Translation between a verbal representation and a pictorial representation.

Description: The student somehow relates a pictorial representation to a verbal representation.

Indicators:

- a. The student makes a pictorial representation with paper and pencil directly from the word problem, either without modifying it or adapting it to his usual style of talk.
- b. The student transforms or modifies a pictorial representation according to a new interpretation of the word problem.
- c. The student establishes relationships between the word problem and a pictorial representation, using verbalizations or gestures.
- d. The student represents pictorial elements using body movements, especially with the hands, while reading the word problem.

Episode 5: Translation between a pictorial representation and a symbolic representation.

Description: The student somehow relates a pictorial representation to a symbolic representation.

Indicators:

- a. The student formulates on paper a symbolic expression or part of one based on a pictorial representation, or makes a pictorial representation based on a symbolic expression.
- b. The student establishes relationships between a symbolic expression and a pictorial expression using verbalizations or gestures.
- c. The student makes changes or eliminates a pictorial representation made earlier, based on symbolic results obtained.
- d. The student modifies or eliminates symbolic representations due to results obtained in pictorial representations or due to a new pictorial representation.
- e. The student assigns symbols to a pictorial representation.

Episode 6: Translation between a symbolic representation and a verbal representation.

Description: The student somehow relates a symbolic representation to a verbal representation.

Indicators:

- a. The student formulates a symbolic expression or part of one based on the word problem, either without modifying it, or adapting it to his usual style of talk.
 - b. The student transforms or modifies a symbolic expression due to a new interpretation of the word problem.
 - c. The student reformulates the word problem in a new way due to some result obtained from a symbolic expression.
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- d. The student assigns a variable to some part of the word problem.
 - e. The student relates a symbolic expression to the word problem through verbalizations or gestures.

Episode 7: Events not classifiable as representation.

Description: Events that are not found in the episodes above.

Indicators:

- a. The student verbalizes planning or execution expressions.
 - b. The student verbalizes or makes gestures of emotional or affective expression.
 - c. The student verbalizes expressions of verification.
-

With this framework for analysis of problem-solving protocols, one of the researchers analyzed the transcriptions and generated a written protocol, which was then checked by another researcher using the same protocol analysis framework. When there was doubt about where to place a certain item within a particular episode, or there was not agreement between the researchers, both of them watched the video again and came to an agreement on what episode a certain item belonged to, thereby increasing the reliability of the analysis.

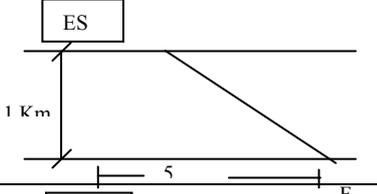
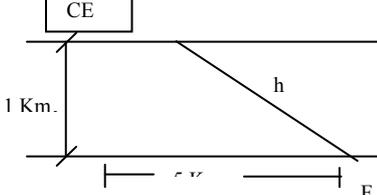
The written work produced by the solver during the problem-solving session was then attached to this protocol. In order to do so, the researcher watched the video again, matching up the verbalizations from the solver with the material written during the problem solving session. We attempted to make this matching as close as possible to what the solver had done during the session, so that, upon observing the document, one “perceives” what the solver was doing at each moment of the process, that is, as you read the verbalizations produced, you know the exact moment when the solver was writing on the paper given to him for problem solving. Along with this, the final transcription was also marked with the time at which each item began and ended.

Finally, we added the gestures that accompanied deictic words (this, here, this way, etc.), the former being very important in order to understand what the solvers were referring to; for this purpose a hand was drawn which pointed to the part of the solution that was being referred to. In order to make the final protocols more readable, we used the following conventions:

1. Text on the left: verbalizations produced by the solvers.
2. Text and graphics on the right: written text produced by the solvers during the problem-solving session.

3. Text between square brackets: actions performed by the solver and interpreted by the researcher.
4. Text in italics: interventions from the researcher.
5. Ellipsis points (...): pauses in the verbalizations produced by the solver.
6. Time (at the beginning and above each item): starting time for each episode. Naturally this time also represents the end time for the previous episode.

Figure 3. Fragment of protocol of solver A solving problem 1.

<p>01:29 OK let's see if we can cross the water</p>	
<p>01:43 ... then the least distance by water would be the hypotenuse,</p>	
<p>01:44 the hypotenuse,</p>	
<p>01:56 we calculate $1^2 + 5^2$, that would be 25 and 1 26 the hypotenuse, is the square root of 26</p>	$h^2 = 1^2 + 5^2 \Rightarrow h = \sqrt{26}$
<p>02:14 well, then ..., if we want to lay it by land ..., then it would be 1 by water this would be one possibility,</p>	
<p>02:32 first possibility by water only,</p>	<p>1° $h^2 = 1^2 + 5^2 \Rightarrow h = \sqrt{26}$</p>
<p>02:35 second possibility eh, we can do it by water and by land, that would be 1 Km by water plus 5 Km by land</p>	<p>2° 1 Km. + 5 Km.</p>
<p>03:24 20. then, the total cost will be 3y euros, 5x euros</p>	<p style="text-align: right;">3y € total cost 5x €</p>
<p>03:52 21. in total that makes $3y + 5x$, when is this function at a minimum?</p>	

$$c = 3y + 5x = f \text{ minimum}$$

Results

Protocols from each participant were analyzed as described in the section above, and afterward results were reported in the form of case analysis. The latter consisted of making an interpretation of the work performed by the solvers. The analyses are presented in narrative form with a series of interpretive comments, and supported by some illustrative items. The material used in order to prepare these included the written protocols drawn up using the framework for protocol analysis developed in this study, the results obtained from analysis of the tables (Villegas, 2002) and a “review” of the written work produced by the solvers. In the case analysis one can observe the solver’s behavior when confronting the problem, his or her competence in using representations, skill in translating between different representations, and how planning, monitoring, control and “local affectivities” (Goldin, 1998a) are either generators of success or creators of obstacles in problem solving. We present below the case analysis of solver C trying to solve problem 1.

Case Analysis: Problem 1 Solver C

The solver begins with a brief reading episode; although he feels uncertain whether to begin solving the problem, “Can I start to write now?”, (2) this episode was not classifiable as a representation and the verbalizations were of an affective type. He continues reading the problem, but is still not doing any translation to another representation. When he finishes reading the word problem, he sees the need to translate the verbal situation to a pictorial one, “I’m going to draw”, (4) and he begins to do the translation.

C makes a pictorial representation directly from the word problem, beginning to represent the situation through the drawing, and verifying the relationship between the “drawing” and the word problem, “width, upstream” (7), simultaneously while verbalizing, he points to the drawing while reading the problem.

Manifesting that he doesn’t understand the problem, C re-reads most of the word problem; when finished he gives signs of knowing how to solve the problem, verbalizing a kind of plan, “this is a combination of land and water, OK, an optimization function”, (9) although what he is speaking of is not an “optimization function”, of course, one deduces that what he

means is an optimization problem. Next, he begins to assign variables to the possible ways of “laying cable” and at the end he re-reads the question formulated in the word problem, “...per meter, What would be the most economical route for the cable to be laid?” (13) the purpose is to be sure he is staying on track.

C poses an equation for the cable route directly from the pictorial representation he had made, “the cable route will be ... $y + x$ meters, that would be minimal” (14) although he sets up the problem incorrectly, forgetting to take into account the price of the cable, he shows skill in translating between the pictorial representation and the symbolic representation.

He immediately relates the equation he has formulated with the word problem, using words from his usual style of talk, “this is what they were asking me, how much is the $y + x$ that gives the minimum distance?” (15) as can be observed, the solver has much ability to translate between representations, moving appropriately between the three representations in a brief period of time (less than 20 seconds).

Next, he poses the need to relate x to y , observing the figure he made previously, “let’s find a relationship between x and y , where x is by land and y is by water» (16) this is not too clear to him, so he relates the figure with the word problem looking for ideas that suggest the way to go, he does not find any, becomes a bit desperate, “... good grief! x by meters, that would be minimal” (18) he realizes that the cables have different prices and constructs the cost function (with the prices backwards, a case of nerves?) and he verbalizes a planning statement “... I have y , if I put x as a function of y , and I optimize, I get a minimum, I calculate how much y costs as a function of x ” (22).

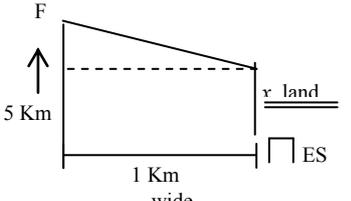
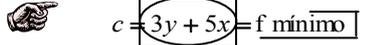
Using the cost function he formulated: he isolates y as a function of x , “I get $y = -5x / 3$ ” (23) in order to do this, did he suppose the cost is zero?, or are nerves playing a fundamental role?

<p>03:24 22. <i>then, total cost will be $3y$ euros, $5x$ euros</i></p>	<p style="text-align: right;">$3y$ € total cost $5x$ €</p>
<p>03:52 23. <i>the total is $3y + 5x$, what is the minimum of this function?</i></p>	<p style="text-align: right;">$c = 3y + 5x = f$ minimum</p>

He manifests uncertainty once again, first recalling, then reading the word problem and also verbalizing relationships between a symbolic expression and the word problem, in a search for two things: a solution to the problem, and a way to calm down.

C derives the function of y , and relates it mentally to a pictorial representation, “where the minimum is a straight line”, (30) “... the minimum would reach ... quickly ... $y' = -\frac{5}{3}$,” (31) by which he commits serious mistakes, which he notices right away, and decides to read the problem more carefully, “I’ll read the problem slowly”, (33) he reads the problem again, and one can observe better concentration in his reading.

After reading the problem, he looks over all that he has done so far, taking into account the figure and the equations, and relating them immediately with the word problem in order to be sure that he is doing it right. After this he takes up the cost function again, and the idea of relating x and y ,

<p>06:36 38. <i>if this is x, this is $5 - x$, y this is y</i></p>	
<p>06:42 39. <i>like this is what I want to be minimal</i></p>	
<p>06:44 40. <i>I'm going to find the relationship between x and y,</i></p>	

The plan which C manifests takes him in the right direction, and shows that, in addition to being skillful in handling representations and translations between them, he has good control and good management in problem solving.

Later, through the use of the figure, he establishes the relationship between x and y , “where I have y as equal to the square root of $1 + 25, 26 + x^2 - 10x$ ” (44) and when he is going to substitute the relationship found in the cost function, he realizes his mistake in mixing up the cable prices, and he corrects them.

<p>08:06 48. then the cost is equal to 3 which means the euros it costs per meter, by land,</p>	<p>by <u>land</u> <u>3 euros / meter</u></p>
<p>08:15 49. [mumbles] we call 3 euros per meter this is by land, no, this would be 3x and by water it costs 5, this would be 5y,</p>	
<p>08:37 50. then I have the cost function of $3x + 5y$ this is what I want to be minimal,</p>	<p>$c = 3y + 5x \leftarrow$ minimum Total cost $\begin{matrix} 3y & \text{€} \\ 5x & \text{€} \end{matrix}$</p>

He substitutes the equation, operates symbolically with tranquility until when he is about to take the derivative, he seems dissatisfied that a square root forms part of the function, “the root could be removed, couldn’t it?” (54) his tranquility wanes, insecurity rises, “anyway this is it, let’s see what comes out”, (55) he takes the derivative of the function, and simplifies it incorrectly, which is not going to affect reaching the solution to the problem.

He continues operating with symbolic representations, following the pattern, “solve, observe and/or check, solve, observe and/or check” (we call this pattern solution-verification), a task which is very important since he immediately checks the steps that he is taking.

Despite the fact that C is making good use of representations and of translation between representations, his lack of concentration and his nervousness mess up calculations which should have been trivial for him. The path to the solution has been complicated by the small errors and C tries to find tactics to move him in the right direction, “this would be the square root, let’s see, let’s see, this is too much from my point of view, I will try to turn it around ... because it has to give y in the equation, I think I am going to isolate y ” (68). He continues with the review-verification guideline mentioned above, which brings him to a solution that, although it differs from the result, is valid and acceptable from several points of view: good use of representations, skill in translation between representations, planning, verification, etc.

Discussion

Similar case analysis was carried out for each of the solvers in their attempt to solve each of the problems. This analysis, together with analysis of the times and usage frequencies for each representation and the review of their written work, all of which was done in a broader study (Villegas, 2002), suggest that the three subjects have a different typology in terms of how they use representations and translations between representations in solving optimization problems; this leads us to think that there are well differentiated typologies of problem solvers. We present below a description of the problem solvers who participated in this study:

Problem-Solver A:

A student who is somewhat competent in using representations and translating between representations, since even though he uses them well, he doesn't easily notice the relationships between different representations.

Invests considerable time in realizing what information is offered by certain translations, certain data, and facts intrinsic to the word problem. He does not quickly realize the most appropriate representation to use at a given moment, and sometimes feels insecure in using it.

Problem-Solver B:

A student who is incompetent in the use of representations and translating between representations in solving optimization problems. Little skill in performing translations with precision and efficiency. Frequently has trouble understanding the word problems, drawing out the information from them and making translations from the word problem to any other representation system.

Even though sometimes he has the information needed to solve the problem, he lacks the necessary skill to uncover in "certain pieces" the keys or necessary information to solve the overall problem. Makes little use of symbolic algebra, and sometimes incorrectly.

Problem-Solver C:

Very competent student in solving optimization problems. Much skill in making translations between different representations with precision and efficiency. A clear understanding of the type of information that a translation reveals. Skill in recognizing the relevancy of the data, relationships and facts expressed in the word problem, or revealed by a translation or representation. Skill in drawing out information from a representation and using it in another type of representation. Skill in working with more than one representation and a capacity to work with more than two representations at the same time.

This student confirms what Lesh, Post and Behr (1987) indicate, “good problem solvers tend to be sufficiently flexible in their use of a variety of relevant representational systems that they instinctively switch to the most convenient representation to emphasize at any given point in the solution process” (p.38).

Conclusions

The purpose of this study was to describe how expert problem-solvers use representations and translation between representations in solving optimization problems. For this purpose, we first developed a theoretical framework for protocol analysis, which is an adaptation of that developed by Schoenfeld (1985), where we take representations and the translation between representations as episodes². In the process of segmentation and coding, we noted that certain segments do not correspond to the episodes identified initially, therefore it was necessary to add in these episodes as “events not classifiable as representations”, comprising verbalizations about planning and execution, affectivity or verification.

Once the framework was developed, it was applied to the verbal protocols generated by the problem-solvers; data analysis suggests the viability and utility of this framework for investigating the influence of representations in solving mathematical problems³. Even so, we are aware that the framework developed for protocol analysis needs a few modifications that would increase its utility.

² Periods of time during which a solver is occupied in a specific action.

³ Despite the fact that this study was done with optimization problems, we believe that the framework is also valid for other types of problems.

Taking the completed analyses into account, we found that the three participants have different typologies in how they use representations and translations between representations in solving optimization problems, this leads us to think that there are well-defined typologies of solvers.

A characterization of the solvers makes evident that there is a strong relationship between success in solving optimization problems and skill in the construction, use and articulation of representations.

We consider that data collection activities were ample and in depth. Review of the written work, the verbal protocols, the video recording, the written protocols, the time and frequency tables, allow us to acquire in-depth understanding of each solver. Likewise, the layout of the written protocols, in addition to giving us further information, makes them “easy” for reading and interpretation, such that other researchers can draw out their own impressions from them. Additionally, the methodology makes it possible to know at each moment which representations and translations between representations are expressed by each solver while attempting to solve optimization problems, and whether he or she is skillful in their use.

We are aware that this study does not provide the basis for establishing typologies of subjects, but it does provide an initial approach for other studies that might be done in this same direction. We feel that more research is needed whereby we can observe the influence of representations in problem solving, especially in those problems where the starting representation is a word problem in natural language.

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