Computational and measurement estimation: curriculum foundations and research carried out at the University of Granada, Mathematics Didactics Department

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Abstract

Researchers from the Mathematics Didactics Department at the University of Granada, members of the research group "FQM193. Number Thinking", have been working since 1985 on estimation, one of the components of number thinking. Our aim in this paper is to offer a global review of the most important contributions from this research group on the topic of estimation. Firstly, we examine some theoretical and curriculum considerations. Secondly, we present a revision of the research literature, and finally, we conclude with a description of some of the studies that have been carried out in our research group.

Resumen

Investigadores del Departamento de Didáctica de la Matemática de la Universidad de Granada que forman parte del Grupo de Investigación "FQM193. Pensamiento Numérico" han estado trabajando desde 1985 en estimación, una de las componentes del pensamiento numérico. En este trabajo se hace una revisión global de las aportaciones más relevantes de este grupo de investigación en estimación; se comienza por una reflexión teórica y curricular, se hace una revisión de la literatura de investigación y se describen algunas de las investigaciones realizadas en el seno del grupo de investigación.

Introduction

Computational and measurement estimation is a mathematical competency that has been incorporated into the mathematics curriculum since the eighties. Due to its novelty, a great deal of research has been carried out since that decade, especially in the United States, in order to clarify the elements that make up estimation and how to include it in the curriculum. In the Mathematics Didactics Department of the University of Granada, within the research group on Number Thinking, we have been working on this topic from a two-fold perspective: curriculum-related reflection based on research results, which we present in the first part, and research contributions on certain aspects which were yet to be developed, in the second part.

Theoretical aspects of estimation

The concept of estimation

The term estimation has many uses and fields of application. It is thus appropriate to begin by specifying the concept of estimation which this paper refers to. Estimation makes a "judgment of what value results from a numeric operation or from the measurement of a quantity, as a function of the estimator's individual circumstances" (Segovia, Castro, Rico & Castro, 1989, p.18)[1]. Two types of estimation appear within this concept:

a) *Computational* estimation, referring to arithmetic operations and to judgments that can be made about their results. Example: an estimate of the result of 2345 multiplied by 52 is 120,000.

b) *Measurement* estimation, referring to judgments made about the value of a certain quantity or the result that would be found by taking a measurement. Two types of magnitudes are differentiated in measurement estimation: continuous and discrete. For example, the case of continuous magnitudes is seen in how we assess someone else's height as compared to our own; the case of discrete magnitudes is exemplified by estimating the number of persons who participate in a political demonstration.

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[1] The concept of estimation, as well as some of the other considerations presented in this paper, are taken from the books *Estimación en Cálculo y Medida* [Computational and Measurement Estimation] (Segovia, Castro, Rico & Castro, 1989) and *Estimación de cantidades discretas. Estudio de variables y procesos* [Estimation of discrete quantities. A study of variables and processes] (Segovia, 1997)

The general concept of estimation includes implicit **characteristics** expressed by Reys (1984) and expanded by Segovia, Castro, Rico and Castro (1989, p.21):

1. It consists of assessing the value of a quantity or the result of an arithmetic operation.

2. The subject who makes the assessment has some information, reference or experience with the situation to be assessed.

3. The assessment is generally done mentally.

4. It is done quickly, using the simplest numbers possible.

5. The value reached is not exact, but close enough for making decisions.

6. The value reached can vary somewhat depending on the person making the assessment.

Estimation and Number Sense

Number Sense is closely related to estimation. In order to clarify ideas about the meaning of Number Sense, the editorial board of *Arithmetic Teacher* devoted an issue to this topic in 1989, with two objectives: to promote discussion on the definition of Number Sense, and to share practical ideas about how to help students better develop their number sense (Thompson & Rathmell, 1989). For Edwars (1984), Number Sense is a form of mental arithmetic and a certain capacity for comparing numbers. For the *NCTM* (1989), development of Number Sense implies:

a) a good understanding of the meaning of numbers,

b) having developed multiple relationships between numbers,

c) recognizing the relative magnitude of numbers,

d) understanding the relative effects of operations with numbers,

e) developing objects of reference for measurements.

For Sowder (1988), Number Sense is a well-organized network of concepts that makes it possible to relate numbers and properties of operations, that provides skill in working with numeric magnitudes. Greeno (1991) refers to important abilities, such as flexibility in mental computation, numeric estimation and quantitative judgment and uses the metaphor of the conceptual "environment" as a model for its analysis and study; the conceptual field is compared to an environment where people learn to live and to interact. He defines it as "cognitive expertise-knowledge that results from extensive activity in a domain through which people learn to interact successfully with the various resources of the domain, including knowing what resources the environment offers, knowing how to find resources and use them in their activities,

perceiving subtle patterns, solving ordinary problems routinely, and generating new insight" (p.170).

Similar to Greeno's line of thinking is the description of Number Sense given by Howden (1989): "Number sense can be described as good intuition about numbers and relationships. It develops gradually as a result of exploring numbers, visualizing them in a variety of contexts, and relating them in ways that are not limited by traditional algorithms" (p.11).

The Ministry of Education and Science, MEC (2006), in Law 1513, Dec. 7, 2006, establishes the minimum teaching requirements for Primary Education, and defines Number Sense as the "domain of reflection on number relationships, and can be expressed in capacities such as: the ability to break down numbers naturally, understand and use the decimal numbering system, use operational properties and the relations between them in order to perform mental computation".

Even though there are differing opinions about what Number Sense is, all of them include *Estimation* as an integral part, or they considerate it to be closely inter-related.

Estimation and Mental Computation

In processes of estimating the result of an operation or the measurement of a quantity, mental computation takes on a central role. This does not mean that estimation can only work with mental computation algorithms; an estimation can be worked out with pencil and paper or with a calculator; but in most situations this is not possible or desirable, and therefore mental computation is required.

Reys (1984, p.548) establishes two characteristics that distinguish mental computation; the first characteristic is that mental computation produces exact answers, and the second is that it makes use of mental procedures without external helps such as pencil and paper. There is no contradiction between the production of exact answers with the approximate values produced by estimation; Reys refers to mental computation, exclusively, without relating it to estimation. When Mental Computation is used in an estimation procedure, there is a previous selection of simple numbers which are to be operated on mentally; it is this choice of numbers which brings

about approximate answers. He gives five reasons to defend the teaching of mental computation at school:

1) it is a pre-requisite to developing written arithmetic;

2) it furthers the knowledge of number structures and their properties;

3) it promotes creativity, independent knowledge and prompts students to figure out large numbers;

4) it contributes to improved problem solving;

5) it is a basis for developing techniques for computational estimation.

Gómez (1994, pp.35-45) expands on the reasons for including mental computation in the compulsory education curriculum: its relationship to estimation is one of those reasons. Thus, a good basis for estimation consists of having adequate skill with mental algorithms that are easy to memorize and quick to use. Segovia, Castro, Rico and Castro (1989) present some methods and Gómez (1994) also offers an extensive, detailed list of methods.

Estimation and simple or rounded numbers

Estimation produces approximate outcomes because in estimation processes, data are transformed or substituted with simple numbers, that is, numbers that are easy to remember and to work with in mental arithmetic operations. There are several ways to produce simple numbers from exact data:

1) Simple numbers by truncation: truncating a number consists of taking only the most significant digits from the front end of the number, as the situation calls for. In order to operate with the simplified number, the eliminated digits can be replaced with zeros when they are integer values (example, a truncation of 3572 would be 3570, or 3500, or 3000); or one can operate with the truncated number as is and add the zeros to the result later on.

2) Simple numbers by rounding: like truncation, rounding a number consists of taking only the most significant digits from the front end of the number, as the situation calls for, with the condition that if the first discarded digit is 0, 1, 2, 3 or 4, then the last digit retained (and all the others) remain as is; otherwise, the last digit retained is increased by one unit. For the previous example, possible roundings of 3572 would be 3570, 3600 and 4000.

3) Simple numbers by substitution: when a piece of data is complicated to work with, it can be substituted by another relatively close number which removes the difficulty. For example, in order to obtain an estimate of the result of 36894:7, 36894 is substituted with 35000, making the division 35000:7, which is easy to carry out mentally.

Estimation and approximation

Approximation is a term often used in numeric computation; it is related to estimation but it is not a synonym (Hall, 1984). Sowder (1989) also analyzes the difference between these two terms. Segovia, Castro, Rico and Castro (1989) define approximation in more detail, as well as its relation to estimation. To approximate is to find a result which is sufficiently precise for a certain purpose. Approximation emphasizes closeness to the exact value and it is totally controllable; one approximates as much as the situation requires. Computation theorems (approximate ones) or the theory of errors, and algorithms with paper and pencil or on the calculator are all tools available for approximation. Estimation takes error into account but in a less precise manner. Sometimes there is no assurance of control. Characteristics 3 and 6 of estimation, cited above, are not true of approximation. Estimation may make use of approximate computation theorems to the extent that these theorems can be applied mentally. In general, the present study does not refer to approximation in the terms that we have just defined. The terms *approximate* and *approximation* will appear frequently in this study, in reference to estimation outcomes, which can be described as approximations of an exact value, or approximate values, like the result of an approximation as defined above; we will also refer to approximation as the process that consists of substituting one number for another close number which is simpler.

Estimation is part of mathematics

A simplistic conception of mathematics associates it with exactness; therefore, estimation might seem out of its domain, or a rather incorrect way of doing mathematics. Analyzing the reasons for the use of estimation demonstrates just the opposite. "the uses of estimation fit the ideals of mathematics, namely, clarity in thinking and discourse, facility in dealing with problems, and consistency in the aplication of procedures" (Usiskin, 1986, p.2).

The reasons that make estimation necessary can be classified into four groups:

1) The impossibility of knowing an exact value; as in the case of computation with a value that is unknown in exact terms, for example, the number of automobiles that travel on a given weekend.

2) The impossibility of exact numerical treatment; for example, when computing with a periodic decimal number.

3) Numerical clarity; the media, for example, use estimations instead of exact quantities in order to make the information clearer and more understandable, as in "150 million pesetas for a school population of 63 thousand students", instead of "148,739,426 pesetas for a school population of 62,879 students".

4) Ease in computation; there are many situations where exactness is not necessary and an approximate answer is sufficient and useful; appropriate rounding and appropriate mental algorithms provide a simple way of obtaining an answer which is accurate enough and useful for making decisions.

Estimation in the school curriculum

Reasons for incorporating estimation in the school curriculum

There are two fundamental reasons for incorporating Estimation into the school curriculum. The first is its practical utility, and the second is to round out students' training. The first reason is addressed in the the previous section, where arguments are given that make Estimation necessary in certain situations. The well-known Cockcroft Report (1985) underscores estimation as a necessary complement for adults' mathematics needs: "Estimation can be considered from different perspectives. One type of estimation allows us to obtain an approximate answer before carrying out a calculation … allowing us to verify whether the outcome from an operation has the correct order of magnitude. A second type of estimation could be defined as the ability to determine whether an answer is reasonable or not. In connection with this is the possibility of estimating different types of measurements, where practical experience and continued use undoubtedly yield the best results" (p.95).

Similarly, the U.S. National Council of Teachers of Mathematics reports that Estimation is useful in school tasks when it is used as a tool to check results, or as a resource in teaching certain topics like measurement ("*An agenda for Action: Recommendations for School Mathematics of the 1980s"*). Also, the NCTM (1991) *Curriculum and Evaluation Standards for Mathematics Education* considers that *"the skills and conceptual structures of estimation enhance the abilities of children to deal with everyday quantitative situations" (p.35).*

With regard to the second reason, Estimation broadens the restrictive view of mathematics which we referred to earlier. Classroom teaching should encompass this two-fold nature of mathematics, the exact and the approximate, and should provide students with activities that let them appreciate in what circumstances it is best to use one or the other. Estimation, in summary, "presents students with another dimension of mathematics; terms such as about, near, closer to, between, and a little less than illustrate that mathematics involves more than exactness." (NCTM, 1991, p.36). Edwards (1984, p.61) and Hope (1989, p.15) state the reason which sums up every kind of argumentation in defense of teaching estimation: Estimation develops Number Sense.

Furthermore, a list of the components involved in Estimation provides a clear idea of the importance of this topic in the school curriculum.

Components involved in computational estimation:

Sowder (1989, p.376) provides the following list:

1) Conceptual components:

1.- The role of approximate numbers.

1.1. Recognizing that numeric approximation is used in computation.

1.2. Recognizing that estimation is a procedure which produces approximate values.

2.- Multiplicity of processes / Multiplicity of answers.

2.1. Accepting more than one process in order to obtain an estimation.

2.2. Accepting more than one value as an outcome of an estimation.

3.- Appropriateness

3.1 Recognizing that the appropriateness of processes depends on the context.

3.2 Recognizing that the appropriateness of estimating depends on the desire to approximate.

2) Technical components:

  1. Processes.

   1.1 Reformulation: Changing the numbers used for the computation.

    - Rounding

    - Truncating

    - Averaging

    - Changing the number expression

   1.2 Compensation: making adjustments during and after the calculation.

   1.3 Translation: changing the structure of the problem.

  2. Answers

   2.1 Determining the correct order of magnitude in an estimation.

   2.2 Determining an acceptable estimation.

3) Relating concepts and techniques:

  1. Able to work with powers of 10.

  2. Understand place value of numbers.

  3. Able to compare numbers by size.

  4. Able to compute mentally.

  5. Knowledge of basic facts.

  6. Knowing properties of operations and their appropriate use.

  7. Recognizing that changing the numbers can change the outcome.

4) Affective components:

  1.- Confidence in one's ability to do mathematics

  2.- Confidence in one's ability to estimate

  3.- Error tolerance

  4.- Recognizing that estimation is useful.

Incorporating estimation into the curriculum

In the United States, interest in the topic of estimation is not recent; NCTM yearbooks from the years 1937, 1960, 1976 and 1978 all publish articles on the topic, and the 1986 yearbook is devoted entirely to estimation. A considerable number of articles have been published in journals specializing in mathematics education over the last 30 years. Despite this

emphasis on incorporating the topic of estimation into the curriculum, and the research that has been undertaken, researchers in general present the teaching situation as being tentative. Even though the topic has been incorporated in the curriculum for years, its treatment is very superficial (Reys, 1984) and limited (Hope, 1986, Johnson, 1979, Trafton, 1986, Sowder & Wheeler 1989). Carlow (1986) shows dissatisfaction with the results from investigating a program where estimation had been included and developed since 1969. Assessment tests carried out by the *National Assessment of Educational Progress (NAEP)* (Carpenter, Coburn & Reys, 1976; Montgomery, 1990) reveal the same.

In Spain, estimation became an explicit part of curriculum plans for Primary Education (MEC, 1992) and Compulsory Secondary Education (Nieto et al., 1989). One of the objectives in Primary Education is to develop the capacity to "form and use personal strategies in estimation, mental computation and orientation, and apply them in solving simple problems". This general objective is broken down into the following objectives for skill development (MEC, 1992, pp. 28-35):

1) Estimating quantities and order of magnitude for the outcome of an operation.

2) Predicting and checking the outcome of operations and problems.

3) Estimating and checking the result of measurements.

4) Using different strategies for an exact or an approximate count.

5) Deciding on the appropriateness of doing exact or approximate computations in a given situation, considering how much error is admissible.

6) Estimating the outcome of a calculation by choosing between several proposed solutions, and assessing whether a certain numeric answer is reasonable or not.

7) Developing personal mental computation strategies with simple numbers.

8) Confidence in one's own abilities and satisfaction in performing mental computation.

9) Understanding the importance of measurements and estimations in daily life.

In the current legislation on minimal requirements for Primary (MEC 2006), estimation appears in reference to computation and to measurement. In Section I, entitled Numbers and Operations, we find that "during this stage of education, the pupil should become fluent in computing and make reasonable estimations, establishing a balance between conceptual understanding and computational competency" and in Section II, on Measurement: Estimation and Calculation of Magnitudes, "Estimating the results of measurements taken (distances, sizes, weights, capacities, etc.) in familiar contexts".

Similarly, in the case of Compulsory Secondary, legislation from December 29th, 2006 refers to estimation in computation and in measurement; for computation, "developing a capacity for estimation and mental computation that can manage results and possible errors", for measurement, "problem solving that involves estimation and computation of lengths, areas and volumes".

Research on estimation and contributions from the Number Thinking group of the University of Granada, Mathematics Didactics Department

Next, we describe the two large categories of estimation research – computation and measurement – and other subcategories within these, giving research examples, and including studies done by the Number Thinking group, which are described later.

The case of computation

In Segovia (1987) and Segovia and Castro (2005), we find a review of research related to computational estimation; these form the basis of research studies carried out at the University of Granada Mathematics Didactics Department. They can be classified into four large groups, where we also include one research review.

Relating estimation with other skills

Bestgen and colleagues (1980) compile information about the attitude of pre-service teachers toward computational estimation. Levine (1982) relates the skill of computational estimation to the number and type of strategies used by college students. Rubenstein (1985) relates computational estimation strategies as identified by Reys (1980) – such as use of "compatible" numbers, rounding, etc. – with other mathematics techniques like multiplication and division, addition and subtraction, and proposes a set of activities from daily life where the these strategies are very appropriate. Sowder (1989) and Sowder and Wheeler (1989) relate estimation to a list of components which were noted above. Lynchard (1989) studies the relationship of estimation to number sense in sixth-grade pupils that have received instruction in estimation.

Case and Sowder (1990) relate computational estimation with mental computation and approximation. Kinkade (1991) examines the performance of eighth-grade pupils in estimation and describes the relationship of attitudes towards mathematics and towards estimation with estimation performance. Gliner (1991) analyzes different variables that can influence performance in computational estimation. Mottram (1996) analyzes the influence of the context on the ability to estimate and on choice of estimation strategies. Albertson (1996) compares the estimation performance of learning-disabled students with that of other pupils.

Hanson and Hogan (2000) study computational estimation skills in university students. De Castro, Castro and Segovia (2002) analyze the influence of number type on solving estimation tasks.

Comparison of teaching methods

Bestgen and colleagues (1980) substantiate that instruction given to pre-service teachers on estimation techniques in problem solving improves their ability to estimate and their attitude toward estimation. Jarret (1980) attests to the effect of three teaching methods on estimation outcomes. Shoen et al. (1981) analyze the effects of different teaching programs on computational estimation with natural numbers in grades 4, 5 and 6, showing improved performance on estimation activities at the level of exercises and problems. Abed (1985) compares the effectiveness of the three teaching methods on estimating the quotient of decimal number divisions. Segovia (1986) also substantiates that children who receive instruction in computational estimation show improved performance as compared to those who do not. Whalen (1989) compares two instructional methods for computational estimation.

Sanfiorenzo (1990) carried out an experimental study where he compares the effectiveness of three teaching methods in estimating with decimal numbers. Chien (1990) attempts to show that the computational estimation skills of pre-service teachers can improve if they use suitable self-study materials. Fernández-Cano (1991) compares two instructional methods, with and without calculators, and analyzes their effects on mental computation and estimation in third-grade pupils.

Strategy identification

Reys et al. (1982) identify computational estimation strategies used by good estimators in grades 7 to 12, and they classify these as Reformulation Processes (*Front-end*, use of *Compatible*

Numbers, Rounding and Truncating), Translation Processes and Compensation Processes. Wyatt (1986) identifies processes used in estimations, reasonableness of estimations, and criteria for reasonableness. Brame (1986) investigated strategies used by secondary pupils who are considered poor estimators. Sowder and Wheeler (1989) analyze responses from children in grades 3, 5, 7 and 9 on estimation tasks from the point of view of strategies and outcomes, and their relationship with an extensive list of components involved in computational estimation, referred to above.

Morgan (1990) shows that the context can prompt development of estimation strategies. Flores, Reys and Reys (1990) identify strategies in computational estimation tasks used by Mexican children within the conceptual framework developed by Reys; their results provide validation of the general theoretical framework used in the research by Reys (1982). Segovia, Castro, Rico and Castro (1989) develop a flow chart which incorporates all the possible strategies described by Reys, making it possible to analyze the process of solving estimation tasks. Shoen, Blume and Hoover (1990) analyze the performance of students from grades 5 to 8 on an estimation test with different formats: multiple choice, order of magnitude, rank and reference point, and so on. Dowker (1992) explores accuracy and estimation strategies used in a sample of mathematics professionals. Reehm (1994) identifies estimation processes in eighth-grade students when estimating problems in numerical and contextual format. Dowker (1992) compares estimation strategies of mathematics professionals with three other groups: accountants, psychology students and students of English as a Foreign Language.

Instruction and evaluation

Levin (1981) proposes a series of estimation techniques based on mental computation, the concept of measurement and the real number. For example, one technique for computing 0.7x0.5 is to mentally represent, based on some quantity that is taken as a unit, 70% of the unit and then 50% of that; this way we determine an approximate answer to the initial quantity expressed. Reys (1984, 1988) offers guidance for teaching and assessing estimation. For instruction, use of strategies should be adapted to the traditional sequence of the curriculum: integers, decimals, fractions and percentages. Assessment should emphasize the mental aspect of estimation; other important aspects were controlling time and the use of different formats for test design. Edwars (1983) develops and implements a program for teaching computational estimation to adults. Edwards (1984) analyzes the causes for failure in teaching estimation: the multiplicity of estimation methods, difficult assessment, and the consideration by students and

even teachers that estimation is an "inferior" way of doing mathematics. Reys, Bestgen, Trafton and Zawojewski (1984) prepare some lessons in computational estimation, they implement them and assess the results. Rubenstein (1985) proposes activities to develop strategies identified by Reys et al. (1980).

Segovia (1986) produced a computational estimation test to assess 6th-grade pupiles who have received some estimation lessons in addition and subtraction; the test includes 21 multiple-choice ítems, 14 are additions and subtractions where the need for "carrying" or "borrowing" in the exact operations varies between once, twice, or not at all; seven are multiplication and division items, intended to observe whether the child is able to develop his or her own techniques based on what has been learned with the other operations. Gossard (1986) evaluates teaching received on estimation by comparing what was taught to what was learned. Markovits (1987) analyes the influence of a short treatment of estimation (one lesson) in sixth- and seventh-graders; the students significantly improved their performance after receiving the instruction. Whiteman (1989) analyzes the influence of a period of estimation instruction on learning strategies and on the skill of estimating. Sliva (1988) produces an estimation test with open-answer items in order to assess computational estimation skill.

Goodman (1991) developed an estimation test to assess pre-service teachers and compare those with higher and lower performance. Bobis (1991) investigated the effect of estimation teaching on the skill of estimating and on developing estimation strategies. Murphy (1992) studies the effect of instruction on the estimating skill. Smith (1993) assesses comprehension of computational estimation strategies for addition and subtraction in pre-service teachers. Floyd (1994) carried out an experimental study in order to assess how two instruction sequences on estimation in computing fractions affected students' estimation performance. Clayton (1996) analyzes the role of error percentage in assessing estimation tasks in different research studies. Heinrich (1999) analyzes skill in the use of estimation strategies before and after a period of instruction.

Estimation and approximate computation in Grades 1-8

In this last group of studies we find "Estimation and approximate computation in Grades 1-8" (Segovia, 1986). This paper affirms that "exploitation of content on approximation-estimation concepts by using and practicing computation techniques, over a relatively brief time that does not interfere with teaching other content, improves students'

performance on computation problems where they are asked for an approximate answer based on mental computation". Three computational estimation lessons are presented based on rounding and the use of mental computing techniques that are developed along the way:

Lesson 1: Exact number and approximate number. Rounding a number. Error in rounding. Types of rounding. Order of approximation in rounding. Comparison of errors.

Lesson 2: The usefulness of estimation. Addition techniques. Subtraction techniques.

Lesson 3: Rounding in sums. Approximate sum techniques. Approximate difference techniques.

An estimation questionnaire was also prepared in order to measure students' performance compared to other students who had not received the intervention (see Appendix I).

Diagram of computational estimation strategies

Within the third group of studies, Segovia, Castro, Rico and Castro (1989, p. 151-152) create a diagram which summarizes the processes identified in work by R. E. Reys, Bestgen, Rybolt, and Wyatt (1982); this diagram (Appendix II) constitutes a good instrument for analyzing possible estimation strategies, and has been used in later research such as in De Castro, Segovia and Castro (2002).

Influence of number type in computational estimation

De Castro, Castro and Segovia (2002), based on results from Levine (1980) and Morgan (1990), put forward the hypothesis that the real difference in difficulty of estimation tasks lies in whether the items involve only integers or decimals greater than one, or whether there are decimal numbers less than one. 53 preservice teachers participated in their research; the Levine test (1982) (Appendix 3), was administered to them after a 10-hour period of instruction on estimation (de Castro, 2001, p.269-315). They reach the conclusion that estimating with decimals less than one is more difficult than with integers or decimals greater than one, confirming their hypothesis.

Their work also explores how operation type influences the outcome, and the estimation strategies used by the subjects, according to the model referred to in the previous section; results from this analysis show that subjects use a variety of strategies, ranging from those that make use of only one process, to estimation strategies that incorporate all the processes.

The case of measurement

Joram, Subrahmanyam and Gelman (1988) distinguish between three generations of research in measurement estimation involving estimation tasks. In the first generation, researchers were interested only in the numerical answers which subjects gave on the estimation tasks. For example, in one study of this type, individuals were asked to estimate the measurement of familiar objects which were not present at the time of estimation, such as the height of a doorway; answers were catalogued as "reasonable", "incorrect" and "correct". Results were observed to improve with the age of participants.

In the second generation, research focused on differences between subjects in their skill in estimating attributes such as weight or height, where the quantities to be estimated were present or absent; classification of answers began to include error percentage, although some studies continued to use the classification "correct", "reasonable" and "incorrect". Accuracy improved with age, and length estimations were more accurate than those of weight, capacity or volume.

The third generation of research takes interest in the cognitive processes involved in estimating as well as in the different strategies used.

Segovia (1996) offers a classification which takes into account whether the quantity being estimated is continuous or discrete; the research literature for each case is reviewed below.

Continuous magnitudes

Corle (1963) analyzes pre-service teachers' estimations of length, time, weight, etc. Hildreth (1983) identifies strategies used by 5th-graders, 7th-graders and college students on estimation tasks of length and area. Siegel, Goldsmith and Madson (1985) analyze children's skills in estimating quantities and propose a model that collects the different measurement estimation strategies. Bright (1979) analyzes the influence of practice on estimating linear quantities, showing it to be significant; however, the influence of physical estimation on symbolic estimation (using non-conventional units of measure) is not significant. Markovits (1987) analyzes students' answers to estimation tasks. Clayton (1988) shows that estimations have a 20% error for quantities of less than 100.

Forreste (1990) studies the role of context in estimation in primary school children (5- to 8-year-olds), in estimation tasks involving distance, area and volume. Callis (2002) carries out a study that contributes new understanding on the skill of measurement estimation. Its objectives involve both detecting procedures and resources and strategies applied, at the same time analyzing their frequency of use and effectiveness (accuracy level), and also how these mental estimation structures are organized and generated. Castillo (2006) drew up a questionnaire for estimating quantities of length, surface area, capacity and weight, and finds large estimation errors; he also describes subjects' strategies in estimating.

Discrete magnitudes

In this case we distinguish between estimations that require a numeric answer, and those involving comparison (relative numerosity):

1) With a *numeric answer:* Siegel, Goldsmith and Madson (1985) propose a model for developing estimation strategies which includes numerosity. Crites (1989) identifies estimation strategies, as well as their effectiveness, as used by children in third, fifth and sixth grades. Barody and Gatzke (1991) study the relationship of estimations in preschool children with the actual values of the quantities being estimated: for sets of more than eight elements, estimations show an error greater than 25%; instruction in estimation can be profitable when using quantities smaller than ten. Lentzinger, Rathmell and Urbatsch (1986) show that visual perception in six-year-olds is a limitation on estimation: at this age the quantity of thirty and the quantity of five thousand are similar; both are very large. In order to improve estimation performance, they suggest encouraging use of comparison strategies with known quantities, partitioning into known quantities, and using mental computation. Krueger (1972) relates estimation answers with actual quantities (stimulus); visualization times are very short; perception of the quantity is analyzed exclusively in intervals of seconds.

Karkovits and Hershkowitz (1993) identify estimation strategies used by third-graders and their influence in carrying out estimation activities. Segovia (1997) identifies estimation strategies used by first- to eighth-graders (ages 6 to 14) and analyzes how solution of estimation tasks evolves with age. Pareja (2002) applies the same questionnaire as Segovia (1997) with pre-service teachers in order to identify execution strategies and analyze errors.

2) *Relative numerosity:* Piaget and Szeminska (1964) assert that length influences children's judgments about numerosity. McLaughin (1981) shows that children under 7 years of age involve length and density in their comparison judgments. According to Cuneo (1982), children at an early stage integrate length and density additively in numerosity estimation tasks, and later they do so multiplicatively. For Cowan (1987), estimations by six-year-olds based on counting are inconsistent. Howe and Jund (1987) show us that symmetry in the configuration of the quantity influences estimation outcomes.

Estimation in discrete quantities

Research by Segovia (1997) explores and analyzes mental processes used by six- to fourteen-year-old children in solving estimation tasks with discrete quantities, and their outcomes, as well as how the outcomes and processes evolve with age. The tasks consisted of estimating the quantity of objects (little circles) that appear on a computer screen during a limited time that does not allow subjects to count them; objects were displayed in various geometric structures and had different sizes; answers were typed on the keyboard; both the response and response time were recorded. Additionally, the subjects were asked what procedure they used in making their estimate on four of the 16 tasks which made up the test.

Accuracy of answers evolved with age, with error greater than 50% for the first-graders, about 30% in second- and third-graders, about 20% in fourth-, fifth-, and sixth-graders, and about 15% in seventh- and eighth-graders; accuracy of responses depended on the size of the quantity, the larger the quantity the lesser the accuracy; the structure of the quantity influenced accuracy of the estimation; 12 estimation strategies were identified and grouped into different categories; these strategies were also associated with different age levels. A definition of each of the twelve strategies is presented below, together with the coding which we assigned to it. These strategies fall into four general categories.

1) *Not justified*

Strategy 1: Not justified. Here we include procedures where the child is not able to give any explicit argument, or says something like "I don't know".

2) *Global assessment with no referent.* These strategies are characterized by the child considering the quantity presented as an overall entity, that is, no part of the quantity is considered in order to make an assessment of the total. The different strategies involve whether

or not the child makes an assessment by using a numerical sequence, whether he or she counts over the image (real or mental), and whether or not size is taken into account as an assessment criteria.

Strategy 2: Reciting the numerical sequence without considering the quantity presented. The procedure consists of enunciating the numerical sequence without associating the numbers with the elements in the quantity, stopping on a number without any criterion that relates to the quantity presented.

Strategy 3: Reciting the numerical sequence according to the size. This consists of enunciating the numerical sequence, stopping on a number which the subject associates with the numerical or spatial size.

Strategy 4: Assigning a number without considering the quantity. The child expresses a criterion which does not imply any action of cardinal reasoning: "I made it up", "it popped in my head", "I thought it", etc.

Strategy 5: Assigning a number according to the size. The child assigns a large or small number based on the numerical or spatial size of the quantity: "because it was very big", "because it's little", "because there are a lot", etc.

Strategy 6: Counting the real or mental quantity. The child counts the quantity of little circles while the image remains on the screen, and if there is not enough time to finish counting them during this time, he or she continues counting from a mental image.

3) *Strategies that involve a global assessment of the quantity through comparing it with a referent*

Strategy 7: Assigning a number by comparison. In this case the child assigns a number to the quantity by comparing it to another that he has seen in one of the previous tasks.

4) Strategies that involve a partial assessment of the quantity. These strategies can be classified in two sub-groups:

a)   Without breaking down the quantity beforehand:

Strategy 8: Counting one part and estimating based on the size. The child counts as long as the image is on the screen and deduces the total without giving any justification or by giving simple criteria such as "because it is very big", "because it is little", etc.

Strategy 9: Counting one part, estimating the rest and adding. The child counts as long as the image is on the screen, estimates the rest by comparison to the part already counted, and adds the two together. For example, the child says "I said 23 because I counted 13 and I think there are 10 more".

Strategy 10: Iterate one part over the total. The child counts, subitizes or estimates a quantity of elements, for example 5, iterates the length of the same over the total and obtains the answer by doing partial sums, $(..(((5+5)+5)+5)+ ...)+5$.

b) Breaking down the quantity beforehand

In this case the processes are more complex, and are generally made up of three components: defining one part and its relation to the total, determining the number of elements in this part, and recomposition of the total.

Strategy 11: Determining one half and doubling it. The child counts one half, or tries to do so as long as the image is on the screen, determines the part that is left through comparisons such as "more or less the same", "a few more", etc. and obtains the total by adding or multiplying by two.

Strategy 12: Counting one part and multiplying or adding. The child breaks apart the quantity into three or more equal parts and reconstitutes the total by adding or multiplying the number obtained for one part by the number of parts.

Estimation of discrete quantities by pre-service teachers

Pareja (2001) replicates the above research with pre-service teachers, obtaining improvement in accuracy of estimations as well as in strategies used, although no new strategies are added to those described above.

Estimation of continuous quantities: length, surface area, capacity and mass

Castillo (2006) established the following research objectives, with regard to the estimations of pre-service teachers and students in secondary education:

a) Describe and characterize estimation error in quantities of continuous magnitudes, Length, Surface Area, Capacity and Mass/Weight, as estimated by subjects in this sample.

b) Describe and characterize the different estimation processes used by the subjects.

For this purpose, a questionnaire was designed which asked students to perform diverse estimations and to write down what was their procedure and/or reasoning involved in reaching their answers. The questionnaire was composed of four questions:

1. Estimate the length of the teacher's desk (in cm).

2. Estimate the surface area of the blackboard (in m^2).

3. Estimate the capacity of the wastebasket (in liters).

4. Estimate the weight of a chair (in kg).

Table 1 shows the means of estimations given, as well as mean error percentages for students from each group, for each of the estimations.

Table 1. *Mean estimations and mean error percentages*

		Length	Surface area	Capacity	Mass/weight
9th-graders	Real measurement	*139 cm*	*2.69 m^2*	*12 l*	*4.4 kg*
	Estimated meas.	140.77	2.74 m^2	7.29 l	3.2 kg
	% Error	1.27	1.86	-39.25	-27.27
Pre-service teachers	Real measurement	*150 cm*	*3 m^2*	*12 l*	*4.9 kg*
	Estimated meas.	161.65	3.8 m^2	9.27 l	3,44 kg
	% Error	7.77	26.67	-22.75 l	-29.8

For the case of Length and Surface Area magnitudes, for the two groups of students, there is a tendency to overestimate (that is, to estimate above the actual value). In the case of Capacity and Mass, by contrast, the tendency is underestimation (estimating below the actual value). The idea of whether the referent (quantity of a known object used to compare with the quantity being estimated) is present or absent was taken into consideration in the identification and classification of processes used (Bright, 1976).

The following processes were identified: a) No answer (the student gives no response); b) 'Rough guess' (the student explains that he or she "eyeballed it" or made a guess); c) Iterating a referent that was present; d) Iterating a referent that was absent; e) Delimiting (the value falls between two quantities); f) Comparing the quantity to be estimated with an approximately equal referent (present); g) Comparing the quantity to be estimated with an approximately equal referent (absent); h) Comparing the quantity to be estimated with a multiple of a referent (present); i) Comparing the quantity to be estimated with a multiple of a referent (absent); j) Comparing the quantity to be estimated with a divisor or fraction of a referent (present); k) Comparing the quantity to be estimated with a divisor or fraction of a referent (absent); l) Breaking down/Recomposing in equal parts (the quantity is broken down, each part is estimated and a total is obtained); m) Breaking down/Recomposing in one part plus its complement; n) Breaking down/Recomposing in different parts; o) Indirect techniques (use of formulas); q) Readjustment (estimating and then making readjustments on the value obtained).

Conclusions

We have presented a review of estimation from the point of view of the curriculum and of research, highlighting contributions from our working group. The case of research was treated from different perspectives: strategies in computational estimation; effects of instruction on computational estimation, on performance and on estimation strategies for measurements of discrete quantities at different levels; performance and estimation strategies for measurements of continuous quantities. Through these studies and others, as well as from our own experience as teachers, the difficulty of incorporating estimation into the compulsory education curriculum becomes evident; this is an open field for research where we seek to answer such questions as:

What is the most suitable procedure for incorporating estimation into the curriculum? At what levels of compulsory education is estimation work most appropriate, and how? What is the most appropriate way to assess students' productions in estimation?

In order to respond to some of these questions, our working group continues to work in this topic area, basing our work on results from Castillo (2006). We are applying and verifying the effects of an action plan for the secondary classroom, distinct from what is

usually found in text books, which deals with estimating continuous magnitudes in the concrete case of length and surface area.

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Appendix I . Segovia Test

(For each operation, the student must select the best approximation of the result from among five options.)

1. 32125 + 46164= (   )
(a) 50000   (b) 60000   (c) 70000   (d) 80000   (e) 90000

2. 32125 + 40164= (   )
(a) 65000   (b) 70000   (c) 75000   (d) 80000   (e) 85000

3. 52137 + 64215= (   )
(a) 112000   (b) 114000   (c) 116000   (d) 118000   (e) 120000

4. 81215 + 92107= (   )
(a) 160000   (b) 170000   (c) 180000   (d) 190000   (e) 200000

5. 39203 + 29107= (   )
(a) 63000   (b) 65000   (c) 67000   (d) 69000   (e) 71000

6. 34107 + 57209= (   )
(a) 80000   (b) 85000   (c) 90000   (d) 95000   (e) 100000

7. 59206 + 69105= (   )
(a) 110000   (b) 120000   (c) 130000   (d) 140000   (e) 150000

8. 73107 + 68310= (   )
(a) 135000   (b) 140000   (c) 145000   (d) 150000   (e) 155000

9. 37102 + 28015 + 16007= (   )
(a) 75000   (b) 77000   (c) 79000   (d) 81000   (e) 83000

10. 19106 + 29001 +16003= (   )
(a) 60000   (b) 65000   (c) 70000   (d) 75000   (e) 80000

11. 62121 + 83003 +93111= (   )
(a) 233000   (b) 235000   (c) 237000   (d) 239000   (e) 241000

12. 60121 + 81321 + 91107= (   )
(a) 200000   (b) 210000   (c) 220000   (d) 230000   (e) 240000

13. 76325 – 44103= (   )
(a) 25000   (b) 30000   (c) 35000   (d) 40000   (e) 45000

14. 59763- 21212= (   )
(a) 31000   (b) 33000   (c) 35000   (d) 37000   (e) 39000

15. 71875 – 19621= (   )
(a) 30000   (b) 40000   (c) 50000   (d) 60000   (e) 70000

16. 57645 – 39134= (   )
(a) 15000   (b) 20000   (c) 25000   (d) 30000   (e) 35000

17. 84643 – 77132= (   )
(a) 5000   (b) 6000   (c) 7000   (d) 8000   (e) 9000

18. 4018 x 6= (   )
(a) 23000   (b) 24000   (c) 25000   (d) 26000   (e) 27000

19. 4915 x 4= (   )
(a) 16000   (b) 17000   (c) 18000   (d) 19000   (e) 20000

20. 4213 x 4= (   )
(a) 15000   (b) 16000   (c) 17000   (d) 18000   (e) 19000

21. 4312 x 7= (   )
(a) 28000   (b) 29000   (c) 30000   (d) 31000   (e) 32000

22. 295 x 406= (   )
(a) 80000   (b) 90000   (c) 100000   (d) 110000   (e) 120000

23. 59347 : 3= (   )
(a) 10000   (b) 20000   (c) 30000   (d) 40000   (e) 50000

24. 34568 : 7= (   )
(a) 4000   (b) 5000   (c) 6000   (d) 7000   (e) 8000

25. 83745 : 19= (   )
(a) 5000   (b) 6000   (c) 7000   (d) 8000   (e) 9000
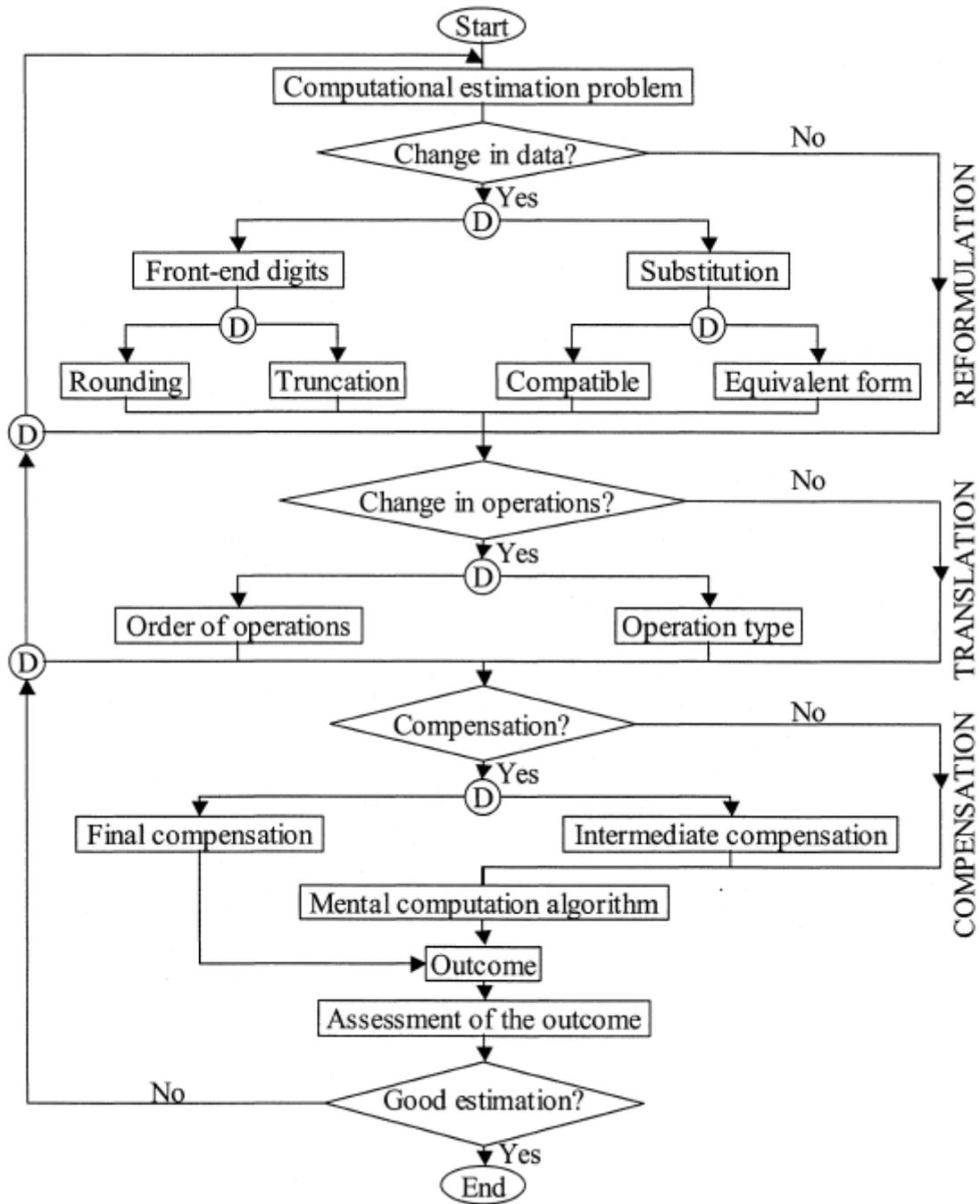
26. 42750 : 19= (   )
(a) 1000   (b) 2000   (c) 3000   (d) 4000   (e) 5000

27. 43000 : 21= (   )
(a) 5000   (b) 4000   (c) 3000   (d) 2000   (e) 1000

Appendix II

Start

Computational estimation problem

Change in data? — No

Yes
D

Front-end digits   Substitution

D   D

Rounding   Truncation   Compatible   Equivalent form

REFORMULATION

D

Change in operations? — No

Yes
D

Order of operations   Operation type

TRANSLATION

D

Compensation? — No

Yes
D

Final compensation   Intermediate compensation

Mental computation algorithm

Outcome

Assessment of the outcome

No — Good estimation?

Yes

End

COMPENSATION

Appendix III

Levine Test

76 x 89

93 x 18

145 x 37

824 x 26

187.5 x 0.06

482 x 51.2

64.6 x 0.16

424 x 51.2

12.6 x 11.4

0.47 x 0.26

9208 : 32

4645 : 18

7858 : 51

25410 : 65

648.9 : 22.4

546 : 33.5

1292.8 : 71.2

66 : 0.86

943 : 0.48

0.76 : 0.89

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